STUDY OF WEAKLY PERTURBED SUPERSONIC FLOWS WITH AN ARBITRARY NUMBER OF NONEQUILIBRIUM PROCESSES

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Plane and axisymmetrical supersonic nonequilibrium flows close to an equilibrium homogeneous flow have been the subject of many papers. Vincenti [1], Moore and Gibson [2], Stakhanov and Stupochenko [3], Clark [4], Der [5], and Ryhming [6] investigated flow past a plane wall and a profile. Morioka and Murasaki [7 and 8] considered plane and axisymmetrical supersonic jets. Interesting results on flow past slender pointed bodies of arbitrary (including axisymmetrical) cross section were obtained by Clark [9]. Slender pointed solids of revolution were also considered by Tkalenko [10] and Khodyko [11]. Napolitano [12], who did not solve actual flow problems, established important relationships between certain thermodynamic and kinetic characteristics of the medium (e.g. between different speeds of sound) and derived equations for the velocity potential.

The aforementioned authors determined flow parameters on the surface of a profile and a slender solid of revolution, on the axis of a plane stream, and on the characteristic extending from the front point of the solid, developed integral representations of the flow parameters, and investigated the damping of perturbations at large distances from the profile, in the region between the initial frozen and equilibrium characteristics. However, all of the authors, except Tkalenko and Napolitano, limited themselves to a single nonequilibrium process. The case of an arbitrary number of nonequilibrium processes is the subject of the present paper.

1. Let us consider the supersonic steady flow of a nonviscous and thermally nonconductive gas in which nonequilibrium physico-chemical processes are occurring. The enthalpy h of a unit mass of the gas is determined by the pressure p , the density ρ (or the temperature T of the translational degrees of freedom of some component of the gas), and by n parameters $q\,(q_1,\ldots,q_n)$, e.g. by the partial masses of the components and by the energies of the various degrees of freedom. We shall investigate plane and axisymmetrical flows close to a homogeneous equilibrium flow proceeding from left to right. Let the direction of the x-axis of the rectangular coordinate system x, y coincide with the direction of unperturbed flow;

in the axisymmetrical case the x-axis is the axis of symmetry and the origin of wall curvature; the leading edge of the profile or the nose of the solid of revolution are situated at $x = 0$. There are no perturbations for $x = -\infty$. The equations describing the flow are of the form

$$\rho u u_x + \rho v u_y = -p_x, \qquad \rho u v_x + \rho v v_y = -p_y$$
$$(\rho u y^\nu)_x + (\rho v y^\nu)_y = 0, \qquad u(2h + w^2)_x + v(2h + w^2)_y = 0 \qquad (1.1)$$
$$u q_{ix} + v q_{iy} = \tau_i^{-1} \omega_i(p, \rho, q) \qquad (i = 1, \ldots, n)$$
$$h = h(p, \rho, q), \qquad w^2 = u^2 + v^2$$

Here u, v are the projections of the velocity of the x- and y-axes; the subscripts x and y denote the corresponding partial derivatives; $\nu = 0$ and 1 in the plane and axisymmetrical cases; the expressions for h and ω_i are known, and in the case of equilibrium all $\omega_i = 0$; $\tau_i \geq 0$ (the relaxation times) are constants inversely proportional to the rate constants of the physico-chemical processes; if $\tau_i = \infty$, then q_i is frozen; if, on the other hand, $\tau_i = 0$, then q_i is in equilibrium and is determined by the equation $\omega_i = 0$.

All of the quantities are dimensionless. Let u_∞° and ρ_∞° be the dimensional velocity and density of unperturbed flow, and l° a quantity with the dimension of length. Reduction to dimensionless form can be effected by dividing x and y by l°, the velocities by u_∞°, the density by ρ_∞°, the pressure by $\rho_\infty^\circ u_\infty^{\circ 2}$, the enthalpy by $u_\infty^{\circ 2}$, the specific entropy s by R , and the temperature by $R^{-1} u_\infty^{\circ 2}$, where R is the gas constant of one of the components. Reduction to dimensionless form of the parameters q_i can be effected by taking account of their dimensions, and this renders the constants τ_i dimensionless as well. In problems not having a characteristic linear dimension $l^\circ = u_\infty^\circ \tau_k^\circ$, where τ_k° is the dimensional value of the relaxation time of the kth process. In this case $\tau_k = 1$.

System (1.1) must be supplemented by relations on the discontinuity surfaces. Let $\tan\theta = v/u$, and let σ be the angle between the discontinuity surface and the x-axis. They can then be written as

$$[\rho w \sin(\sigma - \theta)] = 0, \qquad [w \cos(\sigma - \theta)] = 0 \qquad (1.2)$$
$$[p + \rho w^2 \sin^2(\sigma - \theta)] = 0, \quad [2h + w^2] = 0, \qquad [q_i] = 0 \qquad (i = 1, \ldots, n)$$

where $[\zeta]$ is the difference in ζ at the discontinuity.

2. Linearization of (1.1) and (1.2) is effected in the usual manner. Representing each parameter as the sum of its unperturbed value and a small addend, and retaining the same notation for the addends u, v, p, ρ, h and q as for the parameters themselves, we obtain in place of (1.1) the expression

$$u_x = -p_x, \quad v_x = -p_y, \quad \rho_x + u_x + v_y + \nu v y^{-1} = 0, \quad h_x = -u_x \qquad (2.1)$$
$$\tau_i q_{ix} = \omega_{ip} p + \omega_{i\rho} \rho + \sum_{j=1}^{n} \omega_{ij} q_j \quad (i = 1, \ldots, n), \qquad h = h_p p + h_\rho \rho + \sum_{j=1}^{n} h_j q_j$$

Here for $\zeta = \zeta(p, \rho, q)$ we have introduced the notation

$$\zeta_p = \left(\frac{\partial \zeta}{\partial p}\right)_{\rho,\, q}, \quad \zeta_\rho = \left(\frac{\partial \zeta}{\partial \rho}\right)_{p,\, q}, \quad \zeta_i = \left(\frac{\partial \zeta}{\partial q_i}\right)_{p,\, \rho,\, q_j \neq q_i}$$

The derivatives are here computed for unperturbed flow.

Linearizing (1.2) we obtain

$$[u] \tan \sigma - [v] + [\rho] \tan \sigma = 0, \quad [u] + [v] \tan \sigma = 0$$
$$2\,[u] \sin^2 \sigma - [v] \sin 2\sigma + [\rho] \sin^2 \sigma + [p] = 0 \qquad (2.2)$$
$$[h + u] = 0, \qquad [q_i] = 0 \qquad (i = 1, \ldots, n)$$

The three first equations of this system together with the fourth, rewritten in the form

$$[u] + h_p\,[p] + h_\rho\,[\rho] = 0$$

form a system of linear homogeneous equations for $[u]$, $[v]$,,$[p]$ and $[\rho]$. The slope of the discontinuity surfaces is determined by the condition of its nontrivial solution and is given by

$$\cot \sigma = \pm \beta_\infty \equiv \pm \sqrt{M_\infty{}^2 - 1} \qquad \left(M_\infty{}^2 = c_\infty{}^{-2} \equiv \frac{1 - h_p}{h_\rho}\right) \qquad (2.3)$$

Here c_∞ is the frozen speed of sound divided by u_∞° , so that M_∞ is the frozen Mach number. Thus, as in ordinary gas dynamics, the weak discontinuity lines coincide with the Mach lines of unperturbed flow.

Further, from the first, fourth and last equations of (2.1) with allowance for (2.2) and for the fact that the flow is unperturbed for $x = -\infty$, we find that

$$p = h = -u, \qquad \rho = -M_\infty{}^2 u - \sum_{j=1}^{n} a_j q_j \qquad (a_j = h_j h_\rho{}^{-1}) \qquad (2.4)$$

everywhere.

This and the second equation of (2.1) imply the potentiality of the flow.

If φ is the potential, then

$$u = \varphi_x, \qquad v = \varphi_y \qquad (2.5)$$

The equations for φ and q result from the remaining equations of (2.1) and are of the form

$$\beta_\infty{}^2 \varphi_{xx} - y^{-\nu} (y^\nu \varphi_y)_y = -\sum_{j=1}^{n} a_j q_{j_x} \qquad (2.6)$$
$$\tau_i q_{ix} = -\varkappa_i \varphi_x + \sum_{j=1}^{n} \varkappa_{ij} q_j \qquad (i = 1, \ldots, n)$$

Here the constants $\varkappa_i$ and $\varkappa_{ij}$ are given by the relations

$$\varkappa_i = M_\infty{}^2 \omega_{i\rho} + \omega_{ip}, \qquad \varkappa_{ij} = \omega_{ij} - a_j$$

The nonvortical character of the flow is in line with the lack of an increment s in the specific entropy. In fact, since $s = s(p, h, q)$, it follows that

$$s = \left(\frac{\partial s}{\partial p}\right)_{h,\, q} p + \left(\frac{\partial s}{\partial h}\right)_{p,\, q} h + \sum_{j=1}^{n} \left(\frac{\partial s}{\partial q_j}\right)_{p,\, h,\, q_i \neq q_j} q_j$$

But taking account of the reduction to dimensionless form

$$(\partial s / \partial p)_{h,q} = -(\partial s / \partial h)_{p,q} = -T_\infty^{-1},$$

and from the condition of thermodynamic equilibrium of the unperturbed flow we find that $(\partial s / \partial q_j)_{p,h,q_i \neq q_j} = 0$. Hence from (2.4) we have it that $s = 0$. It also follows that in this approximation, as in ordinary gas dynamics, discontinuities associated with both increases and decreases in pressure are admissible.

The first equation in (2.6) can be changed into a form not containing q_i and their derivatives. This is achieved by its n-fold differentiation with respect to x and by the elimination of q_i from the resulting equations; moreover, prior to each differentiation the q_{ix} are replaced by their expressions from (2.6). The resulting equation for φ is second order in y and of order $(n + 2)$ in x.

The number of constants in (2.4) and (2.6) is important in obtaining the similarity condition. There are $3n + n^2$ such constants in addition to β_∞ (or M_∞). The replacement of q_i by $a_i q_i$ and τ_i by $\tau_i a_i^{-1}$ and the division of the kinetics equations by the coefficients of q_{ix} reduces their number to $n + n^2$. Further reduction to $n(n + 3)/2$ can be effected through the use of Onsager's relations [13]. In writing the resulting equations in the form (2.6), as the τ_i of each kinetics equation it is convenient to take a quantity which is the inverse of the modulus of the coefficient of φ_x and q_j in its right-hand side which is of maximum absolute value.

3. For $\beta_\infty^2 > 0$, i.e. for $M_\infty > 1$, system (2.6) has two families of real characteristics in addition to the streamlines $y = \text{const}$ on which the last n equations are fulfilled. If the total derivatives with respect to y along them are primed, then the equations of the characteristics are

$$x' \mp \beta_\infty = 0, \quad \mp \beta_\infty u' + v' + \sum_{j=1}^{n} a_j \tau_j^{-1} \Big(\varkappa_j u - \sum_{i=1}^{n} \varkappa_{ji} q_i\Big) + \nu v y^{-1} = 0 \quad (3.1)$$

Here and below the upper (lower) sign corresponds to the characteristics of the first (second) family. The coincidence of the discontinuity surfaces with the characteristics is a consequence of the linear approximation.

Since bodies which perturb the flow are not present with $x < 0$, it follows by virtue of the parabolicity of system (2.6) that the flow remains unperturbed everywhere to the left of the characteristics emerging from a, the front point of the solid. These characteristics can be discontinuity lines, so that perturbations of the flow parameters as they are approached from the right generally differ from zero. The boundary conditions follow from (1.2), (2.3) and (2.5) and are of the form

$$\beta_\infty \varphi_x \pm \varphi_y = 0, \qquad \varphi = 0, \qquad q = 0 \qquad \text{for } x = \pm\beta_\infty (y - y_a) \qquad (3.2)$$

Here the subscript a denotes parameters at the point a, and the condition $\varphi = 0$ follows from the continuity of the potential.

Conditions (3.2) must be supplemented by conditions of: nonleakage at the boundary of the solid; constant pressure at the boundary of the stream flowing into the medium at rest; symmetry along the flow axis (at emergence of a jet and in channel flow) and limited perturbations at infinity. If $y = y^\circ(x)$ is the equation describing the contour of the solid of revolution, profile, or channel wall, and if p_a is the difference between the pressure of the medium into which the flow proceeds and the pressure of unperturbed flow, then these conditions can be written as

$$\varphi_y(x, y_a) \equiv v(x, y_a) = f(x) \equiv dy^\circ(x)/dx, \qquad \varphi_x(x, y_a) \equiv u(x, y_a) = -p_a = \text{const} \quad (3.3)$$
$$\varphi_y(x, 0) \equiv v(x, 0) = 0 \quad \text{for } x \geqslant \beta_\infty y_a, \quad |\varphi_x|, \ |\varphi_y| < \infty \quad \text{for } x \geqslant \beta_\infty (y - y_a), \quad y \to \infty$$

In accordance with (3.3) and (1.2) the disruption of the continuity of the boundary conditions (e.g. a discontinuity in the contour) at the point b is usually associated with discontinuities at the point in all the parameters except φ and q. The discontinuity propagates along the characteristic which originates at b. The variation of parameter jumps along it can be found from the second equation of (3.1), which is valid on both sides of the discontinuity. Recalling that $[q_1] = 0$, we obtain

$$\mp \beta_\infty [u] + [v]' + [u] \sum_{j=1}^{n} a_j \varkappa_j \tau_j^{-1} + \nu y^{-1} [v] = 0 \qquad (3.4)$$

Writing the second equation of (1.2) in the form

$$\pm \beta_\infty [u] + [v] = 0 \qquad (3.5)$$

and solving the linear equation which results from the elimination of $[v]$ from (3.4) and (3.5), we find that

$$[u] = [u]_b \left(\frac{y_b}{y}\right)^{1/2\nu} \exp\left\{\pm \frac{y - y_b}{2\beta_\infty} \sum_{j=1}^{n} \frac{a_j \varkappa_j}{\tau_j}\right\} \qquad (3.6)$$

Here $[u]_b$ is the discontinuity in u for $y = y_b$. The discontinuities in v, p, ρ and h are proportional to $[u]$ and can be determined from (2.5) and (3.5). Since the perturbation of the equilibrium flow cannot increase without limit with distance from the perturbation source, it must be the case that

$$\sum_{j=1}^{n} a_j \varkappa_j \tau_j^{-1} \leqslant 0 \qquad (3.7)$$

Moreover, in view of the independence of the nonequilibrium processes, the signs of all the terms must coincide. Near equilibrium ω_i are proportional to the partial derivatives of the entropy with respect to q_i, so it can be assumed that (3.7) follows from the conditions of stability of the equilibrium state.

As is evident from (3.6) and (3.7), the perturbation damping rate increases with decreasing τ_j and can be very large. It is interesting that for almost complete damping it is sufficient for the inequality $|y - y_b| \gg 2\beta_\infty \tau_j / |a_j \varkappa_j|$ to be fulfilled for at least one process. This, of course, does not imply the rapid damping of continuous perturbations to the right of the frozen characteristic.

Formula (3.6) indicates that in the axisymmetrical case the perturbations increase to infinity approaching the axis of symmetry even when they are arbitrarily small for finite y as a result of the damping which results from nonequilibrium. Because of the limitations of linear theory this result does not correspond to true flow and requires nonlinear analysis. The same situation obtains in ordinary gas dynamics, the difference being that there the perturbations increase monotonously.

4. Let us investigate the flow in the neighborhood of the initial point a and the characteristic ac (the boundary of the perturbed zone) (Fig.1). In the axisymmetric case for $y_a = 0$ we have a pointed solid of revolution, and for $y_a > 0$ a body of a shape close to that of a cylinder. In the variables

$$r = \sqrt{x^2 + (\eta - \eta_a)^2}, \qquad \vartheta = \cos^{-1}(x/r), \qquad \eta = \beta_\infty y \qquad (4.1)$$

the point a corresponds to $r = 0$, the characteristic ac is the ray $\vartheta = \pi/4$, and the velocity components are given by Formulas

$$u = \varphi_r \cos\vartheta - \varphi_\vartheta r^{-1} \sin\vartheta, \qquad v = \beta_\infty (\varphi_r \sin\vartheta + \varphi_\vartheta r^{-1} \cos\vartheta) \qquad (4.2)$$

We shall attempt to obtain $\varphi(r, \vartheta)$ and $q_i(r, \vartheta)$ in the form of series

$$\varphi(r, \vartheta) = \sum_{k=1}^{\infty} \varphi_k(\vartheta) r^k, \qquad q_i(r, \vartheta) = \sum_{k=1}^{\infty} q_{ik}(\vartheta) r^k \qquad (i = 1, \ldots, n) \qquad (4.3)$$

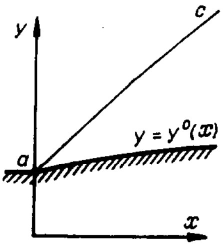

Fig. 1

The boundary conditions (3.2) on ac then become

$$\varphi_k(\pi/4) = q_{ik}(\pi/4) = 0 \quad (i = 1, \ldots, n,\ k = 1, \ldots) \qquad (4.4)$$

The condition of nonleakage at the contour of the body, i.e. the first equation of (3.3), applied to the ray in contact with the body at the point a yields

$$k\varphi_k(\vartheta_a) \tan\vartheta_a + \varphi_k'(\vartheta_a) = \frac{\cos^{k-2}\vartheta_a}{\beta_\infty (k-1)!} y_a^{\circ (k)} \qquad (k = 1, \ldots) \qquad (4.5)$$

where $\vartheta_a = \tan^{-1}(\beta_\infty f_a)$; the primes denote derivatives with respect to ϑ, and $y^{\circ(k)} = d^k y^\circ(x) / dx^k$.

The equations for determining φ_k and q_{ik} are obtained by substituting expansions (4.3) into system (2.6) written out in the variables r and ϑ, and by equating the coefficients of equal powers of r. From the first equation we obtain the recurrent system

$$\eta_a^{\nu}\Big\{k(k+2)\varphi_k \cos 2\vartheta - (2k\varphi_k \sin 2\vartheta + \varphi_k' \cos 2\vartheta)' + \beta_\infty^{-2} \sum_{i=1}^{n} a_i [k q_{ik-1} \cos\vartheta - (q_{ik-1} \sin\vartheta)']\Big\} -$$

$$- \nu \sin\vartheta \Big\{(k-1)\varphi_{k-1}[1 - (k-3)\cos 2\vartheta] + \varphi_{k-1}'[\cot\vartheta + 2(k-2)\sin 2\vartheta] + \varphi_{k-1}'' \cos 2\vartheta - \beta_\infty^{-2} \sum_{i=1}^{n} a_i [(k-1) q_{ik-2} \cos\vartheta - (q_{ik-2} \sin\vartheta)']\Big\} = 0 \quad (k = 1, \ldots) \qquad (4.6)$$

The equations for q_{ik} can be integrated, and with allowance for (4.4) yield

$$q_{ik}(\vartheta) = -\frac{\varkappa_i}{\tau_i}\varphi_k(\vartheta) - \frac{\sin^k\vartheta}{\tau_i} \sum_{j=1}^{n} \varkappa_{ij} \int_{\pi/4}^{\vartheta} \frac{q_{jk-1}(z)}{\sin^{k+1} z} dz \quad (i = 1, \ldots, n,\ k = 1, \ldots) \qquad (4.7)$$

Recalling that $\varphi_k \equiv q_{ik} \equiv 0$ for $k \le 0$, we can show that the solution of the first equation of (4.6) under conditions (4.4) and (4.5) yields the flow of a frozen flow past any cone ($\nu = 1$, $y_a = 0$) or, in the contrary case, of any cone (or obtuse angle). We thus have [14]

$$\varphi(r, \vartheta) = \frac{f_a \sin\vartheta_a}{\beta_\infty \sqrt{\cos 2\vartheta_a}} [\sqrt{\cos 2\vartheta} - \cos\vartheta \operatorname{Arch}(\cot\vartheta)] r + \sum_{k=2}^{\infty} \varphi_k(\vartheta) r^k \qquad (4.8)$$

for $\nu = 1$, $y_a = 0$.

$$\varphi(r, \vartheta) = \beta_\infty^{-1} f_a (\sin\vartheta - \cos\vartheta) r + \sum_{k=2}^{\infty} \varphi_k(\vartheta) r^k \qquad (4.9)$$

for $\nu = 0$ or $\nu = 1$, but $y_a > 0$.

In a small neighborhood of the initial point the flow is determined by the first term, which yields the frozen flow. The remaining terms of the expansion for sufficiently small r do not play a significant role. Due to the finiteness of the nonequilibrium process rates this result is natural and one which has already been proved for bodies of finite thickness [15 and 16].

System (4.6) is too complex to permit analytic construction of the solution for any k . However, the solution can in fact be constructed in the neighborhood of the initial characteristic, since $\vartheta = \pi/4$ is a regular singular point of each equation of the system (with $\nu = 0$ for $k \geq 2$). Methods for constructing linearly independent solutions of the corresponding homogeneous equations already exist [17]. From these and from a particular solution of the nonhomogeneous equation, which is sought in the form of a generalized power series, a solution satisfying (4.4) can be constructed. The successive examination of (4.6) and (4.7) shows that any $\varphi_k(\vartheta)$, and therefore any $\varphi(r, \vartheta)$, is a generalized power series in $(\vartheta - \pi/4)$ with the exponent $^3/_2$ for a pointed solid of revolution and unity in other cases. The first coefficients of each series are proportional to the first coefficient $\varphi_1(\vartheta)$, which is found from (4.8) and (4.9). Upon substitution of expressions for $\varphi_k(\vartheta)$ into (4.3), the series in r which gives the first coefficient of the generalized series for $\varphi(r, \vartheta)$, can be summed. Assuming that the remaining series in positive powers of r and $(\vartheta - \pi/4)$ yield a certain analytic function $\varphi^\circ(r, \vartheta)$, we obtain expressions which are valid near the initial characteristic

$$\varphi(r, \vartheta) = \frac{f_a \sin \vartheta_a}{\beta_\infty \sqrt{\cos 2\vartheta_a}} \left(\frac{\pi}{2} - 2\vartheta\right)^{3/2} r \left[2 + r\left(\vartheta - \frac{\pi}{4}\right)\varphi^\circ(r, \vartheta)\right] \times$$
$$\times \exp\left(\frac{r}{2\sqrt{2}\beta_\infty^2} \sum_{j=1}^{n} \frac{a_j \varkappa_j}{\tau_j}\right) \qquad (4.10)$$

for $\nu = 1$, $y_* = 0$, and

$$\varphi(r, \vartheta) = \frac{\sqrt{2} f_a}{\beta_\infty} \left(\frac{\eta_a \sqrt{2}}{r + \eta_a \sqrt{2}}\right)^{1/2 \nu} \left(\vartheta - \frac{\pi}{4}\right) r \times$$
$$\times \left[1 + r\left(\vartheta - \frac{\pi}{4}\right)\varphi^\circ(r, \vartheta)\right] \exp\left(\frac{r}{2\sqrt{2}\beta_\infty^2} \sum_{j=1}^{n} \frac{a_j \varkappa_j}{\tau_j}\right) \qquad (4.11)$$

in the other cases.

The resulting expressions make it possible to use (4.2) to find the velocity components as functions of r and ϑ and, with the aid of (4.1), as functions of x and y . For a pointed solid of revolution the velocity discontinuities associated with passage through the initial characteristic do not occur. This also happens in ordinary gas dynamics, and was proved for the case under consideration by Tkalenko [10]. On the other hand, (4.10) yields more information than Formula (4.3) in [10], and at the same time proves the validity of the latter at any distance from the axis of symmetry. The derivatives of u and v with respect to ϑ on the initial characteristic are infinite. In the case of plane bodies and bodies of nearly cylindrical shape, u and v are discontinuous by virtue of (4.11), but their derivatives are finite. The discontinuity damping of course coincides with (3.6).

5. In investigating the linearized equations of nonequilibrium flows extensive use has been made of the Laplace transform. In this we can proceed either from system (2.6) or from the $(n + 2)$-th order equation for the potential. Let us follow the first of these alternatives.

In considering problems corresponding to Fig.1, let us take

$$\xi = x - \beta_\infty (y - y_a), \qquad \eta = \beta_\infty y \tag{5.1}$$

instead of x and y as our independent variables.

In the case of a profile or a plane wall, we set, as we did for a pointed solid of revolution, $y_a = 0$. In the remaining problems we take $y_a = 1$, i.e. we choose the ordinate of the point a as our characteristic linear dimension.

In the new variables system (2.6) becomes

$$\begin{gathered} 2\varphi_{\xi\eta} - \varphi_{\eta\eta} + \nu\eta^{-1}(\varphi_\xi - \varphi_\eta) = -\beta_\infty^{-2}\sum_{j=1}^{n} a_j q_{j\xi} \\ \tau_i q_{i\xi} + \varkappa_i \varphi_\xi = \sum_{j=1}^{n} \varkappa_{ij} q_j \qquad (i = 1, \ldots, n) \end{gathered} \tag{5.2}$$

Let s be a complex variable. Recalling that the domain of perturbed flow is given by $\xi \geq 0$, we introduce the representations

$$\Phi(s, \eta) = \int_0^\infty \varphi(\xi, \eta)\exp(-s\xi)\, d\xi, \qquad Q_i(s, \eta) = \int_0^\infty q_i(\xi, \eta)\exp(-s\xi)\, d\xi$$

In using the Laplace transform special consideration must be given the characteristics which are discontinuity surfaces and add extra terms to the expressions for the representations of the derivatives. It turns out, however, that upon application of the Laplace transform to (5.2) and replacement of the derivative representations by their expressions, such terms vanish (by virtue of (3.2),(3.5) and the continuity of φ and q), while Φ and Q are given by Equations

$$\begin{gathered} \eta\Phi'' + (\nu - 2s\eta)\Phi' - \nu s\Phi = s\eta\beta_\infty^{-2}\sum_{j=1}^{n} a_j Q_j \\ \sum_{j=1}^{n} (\varkappa_{ij} - s\tau_i\delta_{ij})\, Q_j = \varkappa_i s\Phi \qquad (i = 1, \ldots, n) \end{gathered} \tag{5.3}$$

Here δ_{ij} is the Kronecker delta and the primes denote derivatives with respect to η. Equations (5.3) make it possible to express all the Q_i in terms of Φ

$$Q_i = sD_i D^{-1}\Phi \qquad (i = 1, \ldots, n) \tag{5.4}$$

$$D = D(s) = \det\|d^{rt}\|, \qquad D_i = D_i(s) = \det\|d_i^{rt}\| \qquad (i, r, t = 1, \ldots, n)$$

$$d^{rt} = \varkappa_{rt} - s\tau_r\delta_{rt}, \qquad d_i^{rt} = d^{rt} \quad (t \neq i), \qquad d_i^{ri} = \varkappa_r$$

Computing the leading terms of D and D_i, which are polynomials of degrees n and $n-1$, we find that for large $|s|$

$$D_i D^{-1} = -\varkappa_i \tau_i^{-1} s^{-1} + o(s^{-1}) \qquad (i = 1, \ldots, n) \tag{5.5}$$

At the same time, without limiting generality we can assume that $D(0) \neq 0$, since the equation $D(0) = 0$ signifies the linear independence of the right-hand sides of the kinetic equations of system (5.2), and therefore makes it possible to reduce their number and the number of parameters q without increasing the order of the remaining equations. As a rule, Equation $D(0) = 0$ indicates that in the choice of q their number exceeded the required minimum, i.e. that finite connections such as conditions stipulating the conservation of chemical elements were not exploited.

Substituting (5.4) into the right side of the first equation of (5.3), we find that

$$\eta\Phi'' + (\nu - 2s\eta)\Phi' - s(\nu + s\eta B)\Phi = 0 \quad \left(B = B(s) = D^{-1}\beta_\infty^{-2}\sum_{j=1}^{n} a_j D_j\right) \tag{5.6}$$

In accordance with (5.5) for large $|s|$

$$B(s) = -s^{-1}\beta_\infty^{-2}\sum_{j=1}^{n} a_j \varkappa_j \tau_j^{-1} + o(s^{-1}) \tag{5.7}$$

Substituting $\Phi = Z\exp(s\eta)$ for (5.6) as in [10], we obtain Equation

$$\eta Z'' + \nu Z' - \eta s^2\sigma^2(s) Z = 0 \qquad (\sigma^2 = 1 + B)$$

whose solutions are expressed in terms of exponentials (for $\nu = 0$) or cylindrical functions (for $\nu = 1$). The coefficients of the linearly independent solutions are determined by boundary conditions (3.3) on the surface of the body and as $\eta \to \infty$. In considering the conditions at infinity we take into account Equation (3.7). The potential φ is found from Φ with the aid of the inverse transform. For flow past the upper surface of the profile we obtain

$$\varphi(\xi, \eta) = -\frac{1}{2\pi i}\int_{s_0 - i\infty}^{s_0 + i\infty} \frac{F(s)}{s\sigma}\exp\{s[(1-\sigma)\eta + \xi]\}\, ds \tag{5.8}$$

Here the integration is carried out along the straight line $\operatorname{Re} s = s_0$ lying to the right of all the singularities of the integrand; $F(s)$ is such that if L^{-1} is the symbol of the inverse transform, then

$$L^{-1}[F(s)] = \beta_\infty^{-1} f(\xi) \equiv \beta_\infty^{-1} dy^\circ(\xi)/d\xi \tag{5.9}$$

Similarly, for a body of nearly cyclindrical shape we have

$$\varphi(\xi, \eta) = \frac{1}{2\pi i}\int_{s_0 - i\infty}^{s_0 + i\infty} \frac{iF(s)\, H_0^{(1)}(is\eta\sigma)\exp[s(\xi + \eta - \beta_\infty)]}{s\sigma H_1^{(1)}(is\beta_\infty\sigma)}\, ds \tag{5.10}$$

where $F(s)$ is determined from (5.9) and $H_0^{(1)}$ and $H_1^{(1)}$ are cylindrical functions of the third kind (Hankel functions).

For a pointed solid of revolution (5.11)

$$\varphi(\xi, \eta) = \frac{1}{2\pi i}\int_{s_0 - i\infty}^{s_0 + i\infty} iF(s)\, H_0^{(1)}(is\eta\sigma)\exp[s(\xi + \eta)]\, ds, \quad L^{-1}[F(s)] = -\frac{\pi[y^{\circ 2}(\xi)]}{4}$$

This formula can be derived from the results of [10]. The integral representations of u and v are found by differentiating (5.8), (5.10) and (5.11) with allowance for (2.5) and (5.1).

In the internal problems we take the ordinate of the point a as our characteristic dimension and define ξ and η as

$$\xi = x + \beta_\infty(y - 1), \qquad \eta = \beta_\infty y \tag{5.12}$$

The boundary conditions are the third equation of (3.3) on the flow axis, and the first or second equation of (3.3) for $y = 1$. The domain of perturbed flow once again lies in $\xi > 0$.

As our final result for flow in a channel we have

$$\varphi(\xi, \eta) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \frac{1}{s} F(s) G(s, \eta) \exp[s(\xi - \eta + \beta_\infty)]\, ds \qquad (5.13)$$

where

$$G(s, \eta) = \begin{cases} \sigma^{-1}\cosh(s\eta\sigma)/\sinh(s\beta_\infty\sigma) & \text{for } \nu = 0 \\ i\sigma^{-1} J_0(is\eta\sigma)/J_1(is\beta_\infty\sigma) & \text{for } \nu = 1 \end{cases} \qquad (5.14)$$

Here $F(s)$ is given by (5.9) and J_0 and J_1 are cylindrical functions of the first kind (Bessel functions).

If by $G(s, \eta)$ and $F(s)$ we understand

$$G(s, \eta) = \begin{cases} \cosh(s\eta\sigma)/\cosh(s\beta_\infty\sigma) & \text{for } \nu = 0 \\ J_0(is\eta\sigma)/J_0(is\beta_\infty\sigma) & \text{for } \nu = 1 \end{cases} \qquad (5.15)$$

$$L^{-1}[F(s)] = -p_a \qquad (5.16)$$

then (5.13) remains valid for a jet as well. The integral representations of u and v are obtained from (2.5) and (5.13) with allowance for (5.12) and (5.14) to (5.16). The representations of q in all cases are determined from the resulting expressions with the aid of (5.4). Instead of the Hankel and Bessel functions in (5.10), (5.11), (5.14) and (5.15) we can make use of the modified Bessel functions I and K , although their arguments are complex for complex s .

6. Integral representations can be used for determining the flow parameter fields. Quite useful here is the multiplication theorem whose use reduces solution to finding the originals of the expressions not containing $F(s)$. Thus, for a plane wall (5.8) and (5.9) yield

$$v(\xi, \eta) = f_a \psi(\xi, \eta) + \sum (f_+ - f_-)_b\, \psi(\xi - \xi_b, \eta) + \int_0^{\xi} f'(t)\, \psi(\xi - t, \eta)\, dt \qquad (6.1)$$

Here b is the contour discontinuity point, f_- and f_+ are the values of $y^{\circ\prime}$ before and after the discontinuity, summation is carried out over all the discontinuity points, and

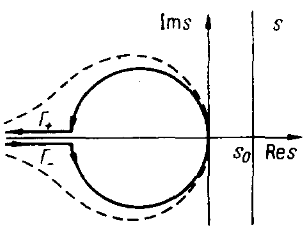

Fig. 2

$$\psi(\xi, \eta) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \frac{1}{s} \exp\{s[(1 - \sigma)\eta + \xi]\}\, ds \qquad (6.2)$$

Similar formulas are obtained in other problems, and not only for v , but for u as well.

The form of $\psi(\xi, \eta)$ depends in large measure on the form of the function $\sigma(s) = \sqrt{1 + B(s)}$. Since $B(s)$ is a ratio of polynomials of degrees $n-1$ and n , it follows that $\sigma^2(s)$ is a ratio of polynomials of degree n . If k° and k are the number of different roots s_j° and s_j of the numerator and denominator, and k_j° and k_j, are their multiplicities, it follows that

$$\sigma^2(s) = \prod_{j=1}^{k^\circ} (s - s_j^\circ)^{k_j^\circ} \Big/ \prod_{j=1}^{k} (s - s_j)^{k_j} \qquad (6.3)$$

The coefficients of the polynomials in the numerator and denominator are real, therefore their roots are either real or complex conjugate, and

$\sigma(\bar{s}) = \overline{\sigma(s)}$. The latter makes it possible to replace the integral in (6.2) by some integral in the upper part of the contour (Im $s \geq 0$), this is valid for any contour symmetrical relative to the real axis. The function $\sigma(s)$ is not single-valued. The single-valued branch is isolated by introducing a number of branch cuts which connect the zeros of the numerator and denominator and lie in the finite portion of the plane s . By virtue of (5.7) the straight line Re $s = s_0$ in (6.2) for $\xi > 0$ can be replaced by a contour consisting of the circle containing all the roots s_j and s_j° and the straight lines Γ_- and Γ_+ (Fig.2). The integrals over Γ_- and Γ_+ cancel, and the contour in (6.2) reduces to the circle. This possibility was already noted by Clark [4]. Turning now to the upper half of the circle and integrating over the real variable, e.g. along the arc of the circle, we represent $\psi(\xi, \eta)$ as a real integral with finite limits which can be evaluated numerically. If all the roots of (6.3) lie in the left half-plane, then the circle can be made to pass through the point $s = 0$ by adding to the integral the contribution of $\frac{1}{2}$ due to the pole at the origin of the coordinate system. Such a contour substitution is also possible in the case of pointed solids of revolution, however, here the parameters on the surface of the body can be found in another way by expanding $H_0^{(1)}$ for small η . As in [10] we have

$$u(x, y^\circ) = \frac{1}{2\pi} S''(x) \ln \frac{y^\circ(x)\beta_\infty}{2} +$$

$$+ \frac{1}{2\pi} \frac{d}{dx} \int_0^x S''(x-t) \left[\ln \frac{\sigma(0)}{t} + \frac{1}{2} \sum_{j=1}^{k} k_j \operatorname{Ei}(s_j t) - \frac{1}{2} \sum_{j=1}^{k^\circ} k_j^\circ \operatorname{Ei}(s_j^\circ t) \right] dt$$

Here $\operatorname{Ei}(z)$ is an integral exponential function, $S(x) = \pi y^{\circ 2}(x)$ is the cross sectional area; it is assumed that $S'(x)$ and $S''(x)$ are continuous (*). The case considered in [10] corresponds to $k_j = k_j^\circ = 1$. For a cone $S(x) = \pi\theta^2 x^2$ and

$$u(x, y^\circ) = \theta^2 \ln \frac{\theta\beta_\infty\sigma(0)}{2} + \frac{\theta^2}{2} \left[\sum_{j=1}^{k} k_j \operatorname{Ei}(s_j x) - \sum_{j=1}^{k^\circ} k_j^\circ \operatorname{Ei}(s_j^\circ x) \right]$$

Since the effect of the nonuniformity must vanish for $x \to \infty$, the two latter formulas imply that all the roots of (6.3) lie in the left half-plane. This property, as well as inequality (3.7) can apparently also be obtained from the conditions of thermodynamic stability.

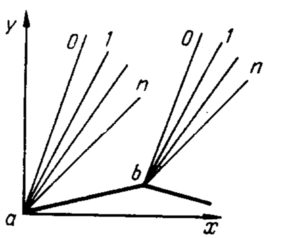

Fig. 3

After u has been determined, the values of q can be found in accordance with (5.4),

$$q_i(\xi, \eta) = \int_0^\xi u(t, \eta)\psi_i(\xi - t)\, dt \qquad (i = 1, \ldots, n)$$

$$\psi_i(\xi) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \frac{D_i(s)}{D(s)} \exp(s\xi)\, ds$$

and are easily be expressed in terms of residues at the points $s_1, \ldots, s_k$.

7. We can use integral representations for determining the properties of the flow far away from the body (for large η). In addition, these can be used to investigate the flow for $\xi = 0$ and $\xi \to \infty$. However, the first case is considered in detail in Sections 3 and 4, while in the second it is

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*) In expression (5.5) in [10], the roots s_j and s_j° in the arguments of E_j are erroneously transposed.

easy to apply the corresponding limiting theorem of operational calculus to show that if there exist constant limiting values of the flow parameters, these are equilibrium in character, i.e. are obtainable from (2.6) for $\tau_1 = \ldots = \tau_n = 0$.

System (2.6) describes flow for any τ . If $\tau_n = 0$, then it can be rewritten, eliminating q_n with the aid of the last equation. The new system differs from (2.6) in the number of equations and in its coefficients which are determined in terms of the same partial derivatives as the initial ones, provided q_n is assumed to be a function of $p, \rho, q_1, \ldots, q_{n-1}$ by virtue of $\omega_n(p, \rho, q) = 0$.

Further, it is possible to set $\tau_{n-1} = \tau_n = 0$, etc. Let us assign subscripts $1, \ldots, n$ to the corresponding M_∞ and β_∞. Here $\beta_{\infty n}$ corresponds to equilibrium flow, $\beta_{\infty n-1}$ to flow in which all q_i are equilibrium with the exception of q_1 , etc.

The results of [12] imply that

$$M_{\infty n} > \ldots > M_\infty, \qquad \beta_{\infty n} > \ldots > \beta_\infty \tag{7.1}$$

Thus, the corresponding characteristics are distributed in the way shown in Fig.3, where the numbers $0, \ldots, n$ denote characteristics with slopes $\beta_\infty, \ldots, \beta_{\infty n}$.

Let us also introduce partially frozen flows with $\tau_1 = \infty$, $\tau_1 = \tau_2 = \infty, \ldots$ and assign the subscripts $1, 2, 3, \ldots, n$ to the functions $B(s)$ and $\sigma(s)$; moreover, $B_n(s) = 0$ and $\sigma_n(s) = 1$. In accordance with these definitions the initial β_∞, M_∞, $B(s)$ and $\sigma(s)$ ought to have been assigned the subscript 0 . It can be shown that

$$\beta_{\infty k} = \beta_\infty \sigma_{n-k}(0) \qquad (k = 0, \ldots, n) \tag{7.2}$$

From (7.1) and (7.2) we find that $\sigma_k(0) > 1$ for $k < n$.

Let us consider flow past a profile. Here we introduce q and τ as stated at the end of Section 2, numbering the parameters q in the order of decreasing τ , and taking $u_\infty{}^\circ \tau_1{}^\circ$ as our l°. The coefficients in $B(s)$ in this case do not exceed unity.

It is natural to expect that far away from the profile the flow is close to equilibrium and that the effect of the initial point to the left of the first equilibrium characteristic is small. To investigate this problem let us consider the behavior of $\psi(\xi, \eta)$ for large y and a $\delta = x - \beta_{\infty n} y$ which is finite or increases more slowly than y .

$$\psi(\delta, y) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \frac{1}{s} \exp(s\delta) \exp\{ys[\beta_{\infty n} - \beta_\infty \sigma(s)]\}\, ds \tag{7.3}$$

By virtue of relation (7.2) the point $s = 0$ is a saddle point of the function $s[\beta_{\infty n} - \beta_\infty \sigma(s)]$. As in the case of a single nonequilibrium parameter [4], for $s = 0$ the contour of steepset descent (the broken curve in Fig.2) touches the imaginary axis, and with increasing distance from $s = 0$ both of its branches asymptotically approach the negative ray of the real axis.

Let $s = 0$ be a unique or the highest saddle point. Replacing the integration contour in (7.3) by the contour of steepest descent and taking account of the residue at the origin as in [4], we obtain

$$\psi(x, y) \sim \frac{1}{2} + \frac{1}{2} \operatorname{erf} \frac{x - \beta_{\infty n} y}{\sqrt{\alpha y}} \qquad \left(\alpha = -\frac{2\beta_\infty{}^2 B_s(0)}{\beta_{\infty n}}\right) \tag{7.4}$$

Here $B_s(s) = dB(s) / ds$, so that

$$B_s(0) = b_1 + b_2\tau_2 + \ldots + b_n \tau_n$$

The constants b_i are on the order of unity. In accordance with the assumption made about the character of the saddle point, $B_s(0) < 0$ and $\alpha > 0$. The difference between (7.4) and the case of a single nonequilibrium process [3 and 4] lies in the form of $B_s(0)$. As in that case, the perturbations between the initial frozen and the equilibrium charcteristics are rapidly damped due to the rapid tendency erf z to -1 , for $z < 0$ and the width of the transitional zone referred to y near the equilibrium characteristic diminishes as $y^{-\frac{1}{2}}$. We note that the condition $y \gg 1$ required for the validity of (7.4) does not exclude very small dimensional values of y with high nonequilibrium process rates.

If $\tau_m \gg \tau_{m+1}$, i.e. with a large difference between the rates of the two groups of processes, the first m are practically frozen, and as regards the remaining ones we can expect a pattern similar to that just considered. Let $\delta = x - \beta_{\infty n-m} y$. We can show that

$$\psi(\delta, y) = \frac{1}{2\pi i} \int\limits_{s_0-i\infty}^{s_0+i\infty} \frac{1}{s} \exp(s\delta) \exp\{ys[\beta_{\infty\, n-m} - \beta_\infty \sigma(s)]\}\, ds =$$

$$= \frac{1}{2\pi i} \int\limits_{s_0-i\infty}^{s_0+i\infty} \frac{1}{s} \exp(s\delta) \exp\{ys[\beta_{\infty\, n-m} - \beta_\infty \sigma_m(s)]\}\, ds + O\left(\frac{y}{\tau_m}\right)$$

After s has been replaced by $s\tau_{s+1}$, the last integral with $y \gg \tau_{s+1}$ reduces to an integral with a large parameter y/τ_{s+1} , which is investigated in the same way as (7.3).

Taking as our characteristic dimension $u^\circ_\infty \tau^\circ_{s+1}$, we obtain

$$\psi(x, y) \sim \frac{1}{2} + \frac{1}{2} \operatorname{erf} \frac{x - \beta_{\infty\, n-m} y}{\sqrt{\alpha y}} \qquad \left(\alpha = -\frac{2\beta_\infty{}^2 B_{ms}(0)}{\beta_{\infty\, n-m}}\right) \tag{7.5}$$

In contrast to (7.4), the accuracy of the latter formula with increasing y increases only for $y \ll \tau_s$. It is subsequently violated for $y \gg 1 > \tau_s$ the flow is described by Formula (7.4). Similarly, for $\tau_1 \gg \tau_2 \gg \ldots \gg \tau_s$ we note a stepwise transition from one partially equilibrium characteristic to another. By virtue of (6.1), all the foregoing is also valid for internal discontinuity points, and translation of the origin to the point under consideration preserves (7.4) and (7.5).

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